J. Fluid Mech. (1996), vol. 320, pp. 287–304

An asymptotic theory for the generation of nonlinear surface gravity waves by turbulent air flow

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(Received 22 November 1994 and in revised form 28 March 1996)

Based on a previous linear theory (van Duin & Janssen 1992), turbulent air flow over a surface gravity wave of finite amplitude is studied analytically by the methods of matched asymptotic expansions and multiple-scale analysis. In particular, an initial-value problem for weakly nonlinear waves is solved, where the initial conditions are prescribed by a single Stokes wave, displacing the water surface. The water is inviscid and incompressible, and there is no mean shear current. Wave–wave interactions are not taken into account. The validity of the theory is restricted to slow waves and small drag coefficient.

We investigate in detail the change of the mean air flow with the evolution of the wave, with a prescribed order of magnitude of the initial wave slope. The rate of change of this flow is fully determined by an evolution equation for the wave slope, which is obtained from the continuity condition for the normal stress at the air–water interface. This equation also determines the amplitude-dependent rate of growth or damping of the wave, for which a closed-form expression is derived. It turns out that nonlinear effects reduce the rate of energy transfer from the mean air flow to the growing wave, which implies that nonlinearity has a stabilizing effect.

For sufficienty large time scales, the slope of the growing wave becomes so large that the original evolution equation, which is approximately a Landau equation, ceases to be valid. For such relatively large wave slope, an alternative evolution equation is derived, which presumably describes the further evolution of the wave until the occurrence of wave breaking. The relative effects of nonlinearity, which can be characterized by a single parameter, increase with increasing wave slope and decreasing wave frequency.

1. Introduction

In 1957 Miles published his classical theory on the generation of surface gravity waves by wind. This theory adequately describes various observed phenomena, and is rather successful in predicting, for instance, the rate at which the generated wave grows. It was readily recognized, however, that this theory is an oversimplification of reality. The effect of turbulence, for instance, is artificially taken into account by the introduction of a logarithmic wind profile. The theory is inviscid, and the direct effects of the turbulence on the wave motions are ignored. Viscous stresses should be incorporated as well, but these are negligible for the gravity-wave regime (Benjamin 1959; Miles 1959). Finally, it is noted that the theory is linear. Thus, a possible change of the wind profile with the evolution of the wave is not taken into account, although this may be important (Fabrikant 1976; Janssen 1982).

Subsequent studies centred on a more realistic modelling of turbulent air flow over a surface gravity wave. This started with the numerical work of Davis (1972, 1974), Townsend (1972), Chalikov (1976, 1978) and Gent & Taylor (1976). Air turbulence was described by an eddy viscosity model. Results of these, and more recent, theories (with turbulence models of varying complexity) were, however, disappointing in that they still showed a substantial disagreement with observed growth rates for both high- and low-frequency waves. In fact, the agreement between Miles' theory and observations is better, which is partially explained in van Duin & Janssen (1992, hereinafter referred to as VDJ). In addition, the results of the growth rates of the generated waves depend on the model of air turbulence, which shows that the problem of proper turbulence modelling is of crucial importance. This seems to be emphasized by more recent studies, see e.g. Al-Zanaidi & Hui (1984), VDJ, Belcher & Hunt (1993) and Burgers & Makin (1993).

Analytical studies, which also include the direct effects of turbulence, are given by Knight (1977), Jacobs (1987), VDJ, Belcher & Hunt (1993) and Belcher, Harris & Street (1994). According to Miles (1993) these studies have in common that they neglect the energy transfer associated with the phase shift across the critical layer (i.e. the layer where the phase velocity of the wave matches the mean air flow), which is essential in the classical Miles theory. As noted by Belcher *et al.* (1994), however, this so-called classical Miles mechanism, which is an inviscid instability mechanism, cannot be effective because of the dominant effects of the turbulent stresses in the critical layer. This is confirmed by numerical studies, which show that the wave stress varies only gradually with height in this layer, (e.g. Townsend 1972), in contrast with the Miles theory, which predicts a discontinuity due to the jump in the phase shift. The incorporation of viscosity does not smooth out this discontinuity sufficiently, cf. the solution of the Orr–Sommerfeld equation for flow at large Reynolds number (Drazin & Reid 1981).

In the present study, we are concerned with the effect of nonlinearity on the generation of surface gravity waves by turbulent air flow. Extending a previous linear theory (VDJ) on this subject, this effect is studied analytically by the methods of matched asymptotic expansions and multiple-scale analysis, applied to a problem with a three-layer structure. We investigate the change of the wind profile with the evolution of the wave. The rate of growth or damping of this wave is determined by an evolution equation for the wave slope, which also determines the rate of change of the wind profile. The validity of the analysis is restricted to slow waves and small drag coefficient. (Part of this paper was presented at the Air-Sea Interface Symposium in Marseilles (1993) and appears in concise form in the proceedings (van Duin 1996).)

To describe the interaction of the turbulence with the wave, we introduce an eddy viscosity model, applied throughout the flow. Similar turbulence models were introduced by e.g. Gent & Taylor (1976), Jacobs (1987), Makin (1989), VDJ and Burgers & Makin (1993). According to Belcher & Hunt (1993), however, such models lead to an incorrect description of the wave-induced Reynolds stresses in the outer layer. Based on results of rapid-distortion theory (Britter, Hunt & Richards 1981), these authors introduce a so-called truncated mixing-length model to describe the small-scale turbulence over the wave. This implies the use of a mixing-length model in a layer close to the water surface, while in the outer layer (away from this surface) the Reynolds-stress gradients are neglected.

Generalization of the Belcher–Hunt model to a nonlinear model may not be straightforward. In particular, the use of rapid-distortion theory implies different turbulence closure schemes for the averaged and perturbation flows, which are not without their own problems. Apart from this, the use of rapid-distortion models awaits further investigation. In the present paper, we therefore maintain the 'classical' eddy viscosity model, which also allows comparison with previous nonlinear studies to test the theory.

In §2 we start from the full Reynolds equations for incompressible air flow, supplied with the proper continuity conditions at the air–water interface. Averaging the equation for horizontal momentum over the dominant wavelength, the wave stress is related to the modified mean air flow. The water is supposed to be inviscid and incompressible, and there is no basic current. Its surface is displaced by a single Stokes wave, with a time-dependent amplitude. It is shown that the initial wave slope should be of the order of ε, where the small parameter ε corresponds to the square root of the drag coefficient at the reference height for the wind speed. In the limit of vanishing wave slope, the air flow profile is logarithmic throughout the flow domain. We also briefly discuss the three-layer structure of the problem. These layers are analysed in detail in the §§3 to 5.

In §6 the results obtained are applied to derive the evolution equation for the wave slope. For sufficiently small wave slope, which implies sufficiently small time scales in case of a growing wave, this equation is approximately a Landau equation, which has an exact solution. By introduction of a suitable scale transformation, we derive an alternative evolution equation, which is valid for larger wave slope and larger time scales than the original Landau equation. Based on these results, we derive in §7 closed-form expressions for the growth-rate parameter, modifed by the wave slope, and the drag coefficient at the reference height for the (varying) wind speed.

Finally, in §8 the results are discussed.

2. Formulation of the problem

2.1. *The air*

The orthogonal frame of reference (x, z) moves with the phase velocity of the wave, measured in the inertial laboratory frame. The x-axis is aligned with the horizontal, unidirectional mean flow; the z-axis points vertically upwards. The air–water interface, when at rest, is located at $z = 0$. The magnitude of the mean flow depends on height. The air is incompressible, and has a constant density ρ. The effect of surface tension is neglected.

The combined effects of molecular viscosity and turbulence are taken into account by assuming that the kinematic viscosity ν is of the form

$$\nu = \nu_0 + \nu_e \, , \tag{2.1}$$

where ν_0 is the constant molecular viscosity, and ν_e is the eddy viscosity, which is time-dependent because of the moving water surface.

In the limit of vanishing wave amplitude the mean flow in the laboratory frame is of the form

$$U = U_0(z) = \frac{u_*}{\kappa} \log\left(\frac{z}{z_0}\right), \tag{2.2}$$

valid for sufficiently large z, where u_* is the friction velocity, κ is the von Kármán constant, and z_0 is the roughness length.

Close to the water surface the velocity profile is assumed to be of the form

$$U_0(z) = \frac{u_*}{\kappa}\{\log(1+\kappa z^+) + BF(z^+)\}, \quad z^+ = \frac{u_* z}{\nu_0}, \tag{2.3}$$

where B is a constant and F is some smooth function, with $F(z^+) = 1 + O(1/z^+)$ as $z^+ \to \infty$. The absence of a mean current in the water implies the condition $F(0) = 0$. For the constant B we take

$$B = -\log(\kappa R), \qquad R = \frac{u_* z_0}{\nu_0}, \tag{2.4}$$

where R is the roughness Reynolds number. For $z^+ \gg 1$ the velocity profile (2.3) is then of the form (2.2).

We introduce dimensionless variables by defining a lengthscale $L = 1/k$, where k is the horizontal wavenumber of the dominant wave, and a velocity scale

$$V = U_0(\alpha L), \tag{2.5}$$

where the constant α is a free parameter of the order of unity.

The coordinates and the water displacement are scaled by L, the velocities by V, the time by L/V, the pressure by ρV^2, and the viscosity by Lu_*. Furthermore, we introduce the parameter

$$\varepsilon = u_*/V, \tag{2.6}$$

which can be regarded as small because it corresponds to the square root of the drag coefficient at the reference height.

In the moving frame of reference, the equations of motion and the continuity equation are of the form

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -c_t - p_x + \varepsilon\left[\frac{\partial}{\partial x}(2\nu u_x) + \frac{\partial}{\partial z}\{\nu(u_z + v_x)\}\right], \tag{2.7}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} = -p_z + \varepsilon\left[\frac{\partial}{\partial x}\{\nu(u_z + v_x)\} + \frac{\partial}{\partial z}(2\nu v_z)\right], \tag{2.8}$$

$$u_x + v_z = 0. \tag{2.9}$$

Here, $\mathrm{D}/\mathrm{D}t$ is the material derivative, the variables in suffix position denote partial differentiation, u and v are the horizontal and the vertical velocity, respectively, and $c = c(t)$ is the phase speed of the slowly modulated wave.

By introduction of the stream function ϕ according to

$$u = -\phi_z, \qquad v = \phi_x, \tag{2.10}$$

one obtains the single equation

$$\left\{\frac{\partial}{\partial t} - \phi_z\frac{\partial}{\partial x} + \phi_x\frac{\partial}{\partial z}\right\}\nabla^2\phi = \varepsilon\left[4\frac{\partial^2}{\partial z\partial x}(\nu\phi_{xz}) + \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2}\right)\left\{\nu\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2}\right)\phi\right\}\right], \tag{2.11}$$

where ∇^2 is the Laplacian.

To investigate the nonlinear interactions with the mean flow, we introduce in the outer layer $z = O(1)$ the decomposition

$$u = U + \tilde{u}, \; v = \tilde{v}, \; p = \bar{p} + \tilde{p}, \; \nu = \kappa z + \tilde{\nu}. \tag{2.12}$$

Here, U is the x-averaged horizontal velocity (the mean flow), $\bar{p}$ is the (x-averaged) modified pressure, κz is the mean eddy viscosity, and the tilde denotes the perturbation part. The mean flow is written as

$$U = -c + U_0 + U_1, \tag{2.13}$$

where U_0 is the dimensionless form of the velocity profile (2.2), with

$$\kappa z U_{0z} = \varepsilon, \tag{2.14}$$

and U_1 is the modified mean flow, induced by nonlinear effects.

Averaging the horizontal momentum equation (2.7) over a wavelength, and making use of (2.12)–(2.14), one obtains the equation

$$U_{1t} = \frac{\partial}{\partial z}\{-\langle \tilde{u}\tilde{v}\rangle + \varepsilon\kappa z U_{1z} + \varepsilon\langle \tilde{v}(\tilde{u}_z + \tilde{v}_x)\rangle\}, \tag{2.15}$$

i.e. the equation for the modified mean flow. The angle brackets denote horizontal averaging.

In the moving frame of reference, the only time scales are those for damping or growth of the wave. In linear theory, the typical time scale proves to be $1/s\varepsilon$ (VDJ), where s is the ratio of the air density to the water density. Thus, the averaged quantities depend on the coordinate z and the 'slow' time $T = s\varepsilon t$. Since s is a very small parameter, the term with the local time derivative in (2.15) may be omitted. Based on a multiple-scale analysis, and making use of (2.14), the equation for the mean flow then integrates to (van Duin 1994)

$$-\langle \tilde{u}\tilde{v}\rangle + \varepsilon\kappa z U_z + \varepsilon\langle \tilde{v}(\tilde{u}_z + \tilde{v}_x)\rangle = \varepsilon^2. \tag{2.16}$$

The amplitude of the water wave is taken to be much larger than the surface roughness length. For the reference frame under consideration, this implies that in the viscous sublayer no balance between the dominant nonlinear terms in equation (2.11) can be obtained. For this reason, we introduce the wave-following coordinate system

$$t_1 = t, \; x_1 = x, \; y = z - \eta_w(x_1, t_1), \tag{2.17}$$

which moves with the surface of the water wave. This curvilinear coordinate system does accommodate the required nonlinear balance because the eddy viscosity now depends on the coordinate y only (§3).

It will be convenient to rewrite the stream function ϕ according to

$$\phi(x, z, t) = \Psi(x_1, y, t_1). \tag{2.18}$$

In deriving the equation for Ψ, we then have to apply the transformation rules

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x_1} - \left(\frac{\partial \eta_w}{\partial x_1}\right)\frac{\partial}{\partial y}, \; \frac{\partial}{\partial z} \to \frac{\partial}{\partial y}, \; \frac{\partial}{\partial t} \to \frac{\partial}{\partial t_1} - \left(\frac{\partial \eta_w}{\partial t_1}\right)\frac{\partial}{\partial y}. \tag{2.19}$$

In the resulting equation, the variables x_1 and t_1 will again be replaced by x and t, respectively.

2.2. *The water*

It is assumed that the flow is inviscid, incompressible (with a constant density), and irrotational. The depth is much larger than the wavelength, and in the laboratory frame no mean current is observed.

The water surface is displaced by a single Stokes wave with a slope of the order of δ, where δ should be related to the small perturbation parameter ε, defined by (2.6).

In van Duin (1994) it is shown that $\delta = \varepsilon$ is the appropriate scaling. Then the water displacement is of the form

$$\eta_w = \varepsilon\eta_1 + \varepsilon^2\eta_2 + O(\varepsilon^3),\ (\eta_1, \eta_2) = (A\mathrm{e}^{\mathrm{i}x} + \text{c.c.}, A^2\mathrm{e}^{2\mathrm{i}x} + \text{c.c.}), \tag{2.20}$$

where c.c. denotes the complex conjugate. Based on the linear theory (VDJ) we find that A depends on the 'slow' times

$$T_n = s\varepsilon^n t, \quad n = 1, 2, 3, \cdots. \tag{2.21}$$

The phase velocity is of the form

$$c = c_0\{1 + 2\varepsilon^2 \mid A^2 \mid + O(\varepsilon^4)\},\ c_0^2 = c_w^2 - s(1 - c_0)^2, \tag{2.22}$$

where $c_w = (gL/V^2)^{1/2}$ is the phase velocity of the linear wave in the limit $s \to 0$. The expression for the phase velocity is correct to $O(s\varepsilon)$, see van Duin (1994).

The hydrodynamic pressure at the surface elevation, denoted by p_w, is obtained by making use of (2.20)–(2.22) and the kinematic boundary condition. Then we obtain

$$sp_w = \varepsilon\left\{(c_0^2 - c_w^2)A\mathrm{e}^{\mathrm{i}x} + \text{c.c.}\right\} + \varepsilon^2\left\{2\mathrm{i}sc_0\frac{\partial A}{\partial T_1}\mathrm{e}^{\mathrm{i}x} + (c_0^2 - c_w^2)A^2\mathrm{e}^{2\mathrm{i}x} + \text{c.c.}\right\}$$
$$+\varepsilon^3\left\{2\mathrm{i}sc_0\frac{\partial A}{\partial T_2}\mathrm{e}^{\mathrm{i}x} + \text{c.c.} + \bar{p}_3 + \text{higher harmonics}\right\} + O(\varepsilon^4), \tag{2.23}$$

where $\bar{p}_3$ determines the modified pressure, which will prove to be $O(\varepsilon^3)$.

We also need the expression for the horizontal water velocity u_w at the surface elevation. This is of the form

$$u_w = -\Psi_{wy} = -c_0 + \varepsilon\{c_0A\mathrm{e}^{\mathrm{i}x} + \text{c.c.}\} + \varepsilon^2\{c_0A^2\mathrm{e}^{2\mathrm{i}x} + \text{c.c.}\} + O(\varepsilon^3). \tag{2.24}$$

At the interface between air and water we require continuity of the tangential velocity (no-slip condition), and the normal stress. In addition, we have to require continuity of the normal velocity at the interface. Making use of (2.10) and (2.17)–(2.19) the kinematic boundary condition reads $\Psi_x = \eta_{wt}$, which implies that $\bar{\eta}_w = 0$, in accordance with (2.20).

2.3. *The three-layer structure*

As noted in VDJ, three layers (an inner layer, an outer layer and an intermediate layer) are needed to obtain valid asymptotic expansions.

For the inner layer, we introduce the scale transformation

$$\zeta = y/\varepsilon_1, \tag{2.25}$$

where ε_1 is the dimensionless molecular viscosity.

In the outer layer, $z = O(1)$, the coordinate z is used initially, but once the solution has been determined, it is rewritten in terms of y in order to accommodate matching with the solution in the intermediate layer.

For the intermediate layer, we introduce the scale transformation

$$\eta = y/\varepsilon, \tag{2.26}$$

where ε is defined by (2.6). From the relation

$$\varepsilon_1 = \frac{\alpha}{R}\mathrm{e}^{-\kappa/\varepsilon}, \tag{2.27}$$

derived from (2.2), (2.4), (2.5) and (2.6), we find that $\varepsilon_1 \ll \varepsilon$. Thus, the intermediate layer is much thicker than the inner layer. Matching of the outer and inner solutions can only be carried out via the solution in the intermediate layer; cf. VDJ.

In fact, there is a fourth layer, namely the critical layer. This is located at the top of the inner layer and at the bottom of the intermediate layer. It turns out, however, that the inner-layer solution can be matched directly with the intermediate-layer solution. Apparently, the critical layer is not important in this respect. The reason for this is that the width and the height of this layer are the same, which implies that viscous effects are dominant. This also applies to the case when the critical layer lies within a sublayer of negligible profile curvature, see e.g. Miles (1962) and Akylas (1982).

3. The inner layer

In the transformation (2.25) for the inner layer, the parameter ε_1 is related to the parameter ε according to (2.27). Thus, the parameter ε_1 is transcendentally small with respect to ε. This admits an inner-layer expansion of the form

$$\Psi_{inn} = \varepsilon_1\{\Psi_\beta(x,\zeta,t;\varepsilon) + \mathrm{TST}\}, \tag{3.1}$$

where TST denotes a transcendentally small term.

The equation for Ψ_β is obtained from equation (2.11) with $v = \varepsilon_1 v_1$ where v_1 depends on ζ only. By application of (2.17)–(2.19) we obtain

$$\mathrm{L}_1\Psi_\beta = K, \quad \mathrm{L}_1 = \frac{\partial}{\partial\zeta}v_1\frac{\partial^2}{\partial\zeta^2}, \tag{3.2}$$

where K is an integration constant, and v_1 is still to be determined.

Introducing the decomposition $\Psi_\beta = \bar{\Psi}_\beta + \tilde{\Psi}_\beta$, both the x-averaged part $\bar{\Psi}_\beta$ and the pertubation part $\tilde{\Psi}_\beta$ satisfy equation (3.2). Taking $K = 0$, we obtain the decoupled equations

$$\mathrm{L}_1\bar{\Psi}_\beta = 0, \tag{3.3}$$

$$\mathrm{L}_1\tilde{\Psi}_\beta = 0. \tag{3.4}$$

Based on (2.3) the mean flow is assumed to be of the form

$$U(\zeta) = -c_0 + \left\{\frac{\varepsilon}{\kappa} + \varepsilon^2 Q + O(\varepsilon^4)\right\}\{\log(1+\kappa\zeta) + BF(\zeta)\}, \tag{3.5}$$

where the 'constant' $Q = Q(t)$ determines the modified mean flow at lowest order. Equation (3.3) implies that the coefficient v_1 in (3.2) is determined from

$$v_1\frac{\mathrm{d}U_0}{\mathrm{d}\zeta} = \varepsilon, \; U_0 = \frac{\varepsilon}{\kappa}\{\log(1+\kappa\zeta) + BF(\zeta)\}. \tag{3.6}$$

The solution of equation (3.4) is written as

$$\left.\begin{aligned} &\tilde{\Psi}_\beta = \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + (\varepsilon^3\log\varepsilon)\Psi_3 + \varepsilon^3\Psi_4 + O(\varepsilon^4\log\varepsilon), \\ &\Psi_n = A_n\zeta + B_n\{(1+\kappa\zeta)\log(1+\kappa\zeta) + \kappa B\int_0^\zeta F(u)\mathrm{d}u\}, \; B_1 = 0, \end{aligned}\right\} \tag{3.7}$$

with $A_n = A_n(x,t)$ and $B_n = B_n(x,t)$. Matching (3.7) with the perturbation part of the intermediate-layer solution (5.1), and applying the no-slip condition at the water surface, these coefficients are uniquely determined.

The intermediate expansion of (3.5) is obtained by writing the asymptotic behaviour as $\zeta \to \infty$ in terms of the intermediate-layer variable (2.26), where use is also made of (2.25) and (2.27). This expansion is determined from the basic expression

$$\log(1+\kappa\zeta) + BF(\zeta) \sim \frac{\kappa}{\varepsilon} + \log\varepsilon + \log(\eta/\alpha) + \text{TST}. \tag{3.8}$$

It is important to note that (3.8) implies that the intermediate expansion of (3.5) is independent of the term $BF(\zeta)$. This implies that the mean flow in the intermediate and outer layers is independent of the flow structure in the inner layer. It is readily shown that this also applies to the perturbation part of the flow. On the other hand, the constant B in (3.6) and (3.7) should be of the order of unity. In view of (2.4) this excludes very rough flow.

4. The outer layer

In the outer layer, the stream function ϕ is written as

$$\phi_{out} = \bar{\phi}(z,t) + \tilde{\phi}(x,z,t), \tag{4.1}$$

where the overbar denotes x-averaging for fixed z.

Substituting (4.1) into equation (2.11), with $v = \kappa(z - \eta_w)$, we obtain the equation

$$-\bar{\phi}_z\nabla^2\tilde{\phi}_x + \bar{\phi}_{zzz}\tilde{\phi}_x + \left\{\tilde{\phi}_x\frac{\partial}{\partial z} - \tilde{\phi}_z\frac{\partial}{\partial x}\right\}\nabla^2\tilde{\phi} = \kappa\varepsilon\left\{(z\bar{\phi}_{zz})_{zz} + \left(z\nabla^2 + 2\frac{\partial}{\partial z}\right)\nabla^2\tilde{\phi}\right.$$
$$\left. -\eta_w\bar{\phi}_{zzzz} + \eta_{wxx}\bar{\phi}_{zz} + \left(\eta_{wxx} - 2\eta_{wx}\frac{\partial}{\partial x} - \eta_w\nabla^2\right)\nabla^2\tilde{\phi} - 2\eta_{wxx}\tilde{\phi}_{xx}\right\}. \tag{4.2}$$

The decomposition (4.1) is expanded according to

$$\left.\begin{aligned} \bar{\phi} &= \bar{\theta}_0 + \varepsilon\bar{\theta}_1 + \varepsilon^2\bar{\theta}_2 + O(\varepsilon^3), \\ \tilde{\phi} &= \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3\theta_3 + O(\varepsilon^4), \end{aligned}\right\} \tag{4.3}$$

where $\bar{\theta}_n = \bar{\theta}_n(z,t)$ and $\theta_n = \tilde{\theta}_n(x,z,t)$, with $\theta_0 \equiv 0$.

Substituting (4.1) and (4.3) into equation (2.16), with $\tilde{v}$ determined by (2.20), we obtain the following hierarchy of averaged equations:

$$\bar{\theta}_{0zz} = 0, \tag{4.4}$$

$$\langle\theta_{1x}\theta_{1z}\rangle - \kappa z\bar{\theta}_{1zz} = 1, \tag{4.5}$$

$$\langle\theta_{1x}\theta_{2z} + \theta_{2x}\theta_{1z}\rangle - \kappa z\bar{\theta}_{2zz} - \langle\kappa\eta_1(\theta_{1xx} - \theta_{1zz})\rangle = 0. \tag{4.6}$$

The equations for the various θ_n, which describe the perturbation part of the flow, are determined from equation (4.2). The interaction between the mean flow and the perturbations are described by equations (4.4)–(4.6) for the various $\bar{\theta}_n$.

The equation for θ_1 is a Laplace equation which implies that, at leading order, the perturbation flow in the outer layer is irrotational. As a solution we take

$$\theta_1 = wA\mathrm{e}^{\mathrm{i}x-z} + \text{c.c.}, \quad w = 1 - c_0. \tag{4.7}$$

The solutions of equations (4.4) and (4.5) are taken as

$$\bar{\theta}_0 = -wz, \quad \bar{\theta}_1 = -\frac{1}{\kappa}z\log z - rz, \quad r = \kappa Q - \frac{1}{\kappa}(1+\log\alpha), \tag{4.8}$$

where Q is the constant in the modified part of the mean flow (3.5).

Making use of (4.7) and (4.8) the equation for θ_2 is obtained, which has a solution of the form

$$\theta_2 = -\frac{A}{\kappa}E_1(2z)e^{ix+z} + F_2 e^{ix-z} + F_3 e^{2(ix-z)} + \text{c.c.}, \tag{4.9}$$

where F_2 and F_3 are constants to be determined, and E_1 denotes the exponential integral (Abramowitz & Stegun 1964).

From (4.7) and (4.9) the first bracketed term in equation (4.6) vanishes identically. The resulting equation has a solution of the form

$$\bar{\theta}_2 = q_0 + q_1 z - 4w \mid A^2 \mid \int_o^z E_1(\zeta)\mathrm{d}\zeta, \tag{4.10}$$

where q_0 and q_1 are constants, which determine the modified flow at higher order.

For the constants F_2 and F_3 in (4.9) and q_0 in (4.10) we take

$$F_2 = \left\{\kappa Q - \frac{1}{\kappa}(\gamma + \log 2\alpha)\right\} A, \quad F_3 = 2wA^2, \quad q_0 = 2w \mid A^2 \mid . \tag{4.11}$$

Then the intermediate expansion of the outer solution reads

$$\begin{aligned}\Psi_{out} \sim & -\varepsilon w\eta - (\varepsilon^2 \log\varepsilon)\frac{\eta}{\kappa} + \varepsilon^2\left\{-\frac{1}{\kappa}\eta\log\eta - r\eta - (wA\eta e^{ix} + \text{c.c.})\right\} \\ & + (\varepsilon^3\log\varepsilon)\left\{4w \mid A^2 \mid \eta + \left(\frac{A}{\kappa}\eta e^{ix} + \text{c.c.}\right)\right\} + \varepsilon^3\Big[-Q\eta\log\eta + \alpha_3\eta \\ & + \left\{\tfrac{1}{2}wA\eta^2 + \frac{A}{\kappa}\eta\log\eta + \frac{A}{\kappa}(-2 + 2\gamma + \log 4\alpha)\eta - \kappa QA\eta - \alpha_2\right\} e^{ix} \\ & - 3wA^2\eta e^{2ix} + \text{c.c.}\Big] + O(\varepsilon^4\log^2\varepsilon),\end{aligned} \tag{4.12}$$

where γ is Euler's constant and α_2 and α_3 are unknown constants. When the term proportional to $\varepsilon^3\log\varepsilon$ in (4.12) is matched with the inner-layer mean flow, we obtain

$$Q = -4w \mid A^2 \mid, \tag{4.13}$$

which determines the modified mean flow; cf. (3.5) and (4.8).

When written in terms of y, we obtain for the outer-layer pressure

$$\begin{aligned}p_{out} \to \varepsilon(-w^2Ae^{ix} + \text{c.c.}) + \varepsilon^2 & \left[\left\{i\kappa + \frac{1}{\kappa}(\gamma + \log 2\alpha) + 4\kappa w \mid A^2 \mid\right\} 2wAe^{ix}\right. \\ & \left. - 3w^2A^2e^{2ix} + \text{c.c.}\right] + O(\varepsilon^3) \quad \text{as } y \to 0.\end{aligned} \tag{4.14}$$

As will be shown, this matches with the pressure at the water surface.

5. The intermediate layer

In the intermediate layer, where the scale transformation (2.26) applies, the stream function ϕ is rewritten in the form (2.18), with $\Psi = \Psi_{int}$ expanded according to

$$\Psi_{int} = \varepsilon\bar{\varphi}_o + (\varepsilon^2\log\varepsilon)\bar{\varphi}_1 + \varepsilon^2\varphi_2 + (\varepsilon^3\log\varepsilon)\varphi_3 + \varepsilon^3\varphi_4 + \cdots, \tag{5.1}$$

where $\varphi_n = \bar{\varphi}_n(\eta, t) + \tilde{\varphi}(x, \eta, t)$ and $\tilde{\varphi}_0 \equiv 0, \tilde{\varphi}_1 \equiv 0$; cf. (4.12).

To derive the equations for the various $\bar{\varphi}_n$, we start from equation (2.7), with $v = \varepsilon\kappa\eta$. Making use of (2.10), and averaging the resulting equations, we obtain

$$\frac{\partial}{\partial\eta}\left(\kappa\eta\frac{\partial^2}{\partial\eta^2}\bar{\varphi}_n\right) = 0, \quad n = 0, 1, 2, 3. \tag{5.2}$$

The equations for the various $\tilde{\varphi}_n$ are derived from (2.11). Making use of the equations $\partial^2\bar{\varphi}_0/\partial\eta^2 = 0$, $\partial^2\bar{\varphi}_1/\partial\eta^2 = 0$, obtained from matching with (4.12), we find

$$\mathrm{L}\tilde{\varphi}_n = 0, \quad n = 2, 3, \tag{5.3}$$

$$\mathrm{L}\tilde{\varphi}_4 = \tilde{\varphi}_{2\eta\eta\eta}\tilde{\varphi}_{2x} + w^2\eta_{1xxx}, \tag{5.4}$$

where

$$\mathrm{L} = \frac{\partial^2}{\partial\eta^2}\left\{\kappa\eta\frac{\partial^2}{\partial\eta^2} - w\frac{\partial}{\partial x}\right\}. \tag{5.5}$$

Based on (4.12) the solutions of (5.2) and (5.3) are given by

$$\bar{\varphi}_o = -w\eta, \tag{5.6}$$

$$\bar{\varphi}_1 = -\frac{1}{\kappa}\eta, \tag{5.7}$$

$$\varphi_2 = -\frac{1}{\kappa}\eta\log\eta - r\eta + (-wA\eta\mathrm{e}^{\mathrm{i}x} + \mathrm{c.c.}), \tag{5.8}$$

$$\varphi_3 = -Q\eta + (\frac{A}{\kappa}\eta\mathrm{e}^{\mathrm{i}x} + \mathrm{c.c.}). \tag{5.9}$$

The solution of equation (5.4) reads (van Duin 1994)

$$\tilde{\varphi}_4 = \left[\tfrac{1}{2}wA\eta^2 + \frac{A}{\kappa}\eta\log\eta + \frac{A}{\kappa}(-2 + 2\gamma + \log 4\alpha)\eta - \kappa QA\eta \right.$$
$$\left. + 2\mathrm{i}A\{-1 + \omega_3(\eta)\}\right]\mathrm{e}^{\mathrm{i}x} - 3wA^2\eta\mathrm{e}^{2\mathrm{i}x} + \mathrm{c.c.}, \tag{5.10}$$

with the solution of the homogeneous part of equation (5.4) given by

$$\omega_3 = \mathrm{i}h\eta^{1/2}K_1(\mathrm{i}h\eta^{1/2}),\ h^2 = -\frac{4\mathrm{i}w}{\kappa}, \quad \mathrm{Im}\, h < 0.$$

The expression for the pressure gradient p_η, obtained from (2.8), is easily integrated with respect to η, where the integration constant is determined by (4.14). Matching the resulting expression with the constant inner-layer pressure, the continuity condition for the normal stress at the air–water interface becomes (van Duin 1994)

$$p_{out} \to p_w + O(\varepsilon^4) \text{ as } y \to 0, \tag{5.11}$$

where p_w is the pressure at the water surface.

6. The evolution equation

The evolution equation will be derived from the continuity condition (5.11) for the normal stress. Thus, matching (4.14) with (2.23), we obtain the equation

$$2\mathrm{i}c_0\frac{\partial A}{\partial T_1} = 2\mathrm{i}\kappa wA + \left\{\frac{2w}{\kappa}(\gamma + \log 2\alpha) + 8\kappa w^2 \mid A^2 \mid\right\}A. \tag{6.1}$$

Equation (6.1) implies a nonlinear frequency shift which is a factor of $1/\varepsilon$ larger than expected. This strong nonlinearity is a consequence of the fact that the modified mean flow in the outer layer is proportional to ε; cf. (4.3), (4.8) and (4.13). This induces the nonlinear term proportional to ε^2 in (4.14), which is matched with (2.23). In the inner layer, on the other hand, the modified mean flow is a factor of $O(\varepsilon)$

smaller, in view of (3.5). As expressed by (3.8) this change in the order of magnitude is an artefact of the logarithmic velocity profile.

It will be convenient to write

$$A = a\mathrm{e}^{\mathrm{i}\psi}, \quad a = \mid A \mid, \tag{6.2}$$

in order to eliminate dependence on the phase ψ. Then equation (6.1) reduces to

$$\frac{\partial a}{\partial t} = \frac{\varepsilon s \kappa w a}{c_0} + O(s\varepsilon^2), \tag{6.3}$$

where we made use of the definition (2.21), with s denoting the ratio of the air density to the water density. At lowest order, this equation is linear and the wave grows or decays exponentially on a time scale $1/s\varepsilon$.

In order to derive the higher-order evolution equation, which should describe the reduction of the rate of growth on a longer time scale due to nonlinear effects, we need to determine the equation for $\partial A/\partial T_2$, combined with (6.1). To that end, the higher-order term in the limit expression (4.14) should be matched with the corresponding term with $\partial A/\partial T_2$ in (2.23). It should be noted that it suffices to consider the first harmonic only. Thus, in what follows, the mean variables and the higher harmonics will be ignored. In addition, we only need to consider the component that is in phase with $\partial \eta_w/\partial x$, because only this will contribute to growth or damping of the wave. The resulting expression for the reduced outer-layer pressure, which contains the 'in-phase component' of the first harmonic only, is then of the form (van Duin 1994)

$$(p_{out})_r \to \mathrm{i}\varepsilon^2 \left\{2\kappa w + \varepsilon(-1 + 2w - 2\gamma - 2\log\alpha - 8\kappa^2 w \mid A^2 \mid)\right\} A\mathrm{e}^{\mathrm{i}x} + \text{c.c.} + O(\varepsilon^4) \tag{6.4}$$

as $y \to 0$.

For the free parameter α in the velocity scale (2.5) we choose

$$\alpha = \mathrm{e}^{-(1+2\gamma)/2} = 0.341 \tag{6.5}$$

to simplify expression (6.4) and the resulting evolution equation.

Matching the term proportional to ε^3 in (6.4) with (2.23), applying the chain rule $\partial/\partial t = s\varepsilon\partial/\partial T_1 + s\varepsilon^2\partial/\partial T_2$, and making use of (6.2) and (6.5), we obtain the higher-order evolution equation for the amplitude of the wave. This is of the form

$$\frac{\partial a}{\partial t} = \frac{s\varepsilon w}{c_0}\left\{(\kappa + \varepsilon)a - 4\varepsilon\kappa^2 a^3\right\} + O(s\varepsilon^3). \tag{6.6}$$

According to equation (6.6), nonlinearity reduces the growth rate of the wave. Furthermore, the conditions for growth or damping of the wave are the same as those for the linear equation (6.3). Thus, when the wave grows, (3.5) and (4.13) imply that the mean flow decreases with increasing wave amplitude. When the wave is damped, on the other hand, the mean flow increases with time. In this respect, the model gives a physically correct description of the interaction of the wave and the mean flow.

When the remainder term in equation (6.6) is omitted, the resulting (or reduced) equation is of the Landau-type and has an exact solution (Drazin & Reid 1981). Furthermore, this reduced equation has a steady solution

$$a = \frac{1}{2(\kappa\varepsilon)^{1/2}} + O(\varepsilon^{1/2}).$$

However, this is not a solution of the complete equation (6.6) because, as shown below, for long time scales the remainder term proves to be of the same order of magnitude as the lowest-order linear term and the nonlinear term in the reduced equation. From the exact solution of the latter equation it is then readily seen that, with the initial condition $a = O(1)$ at $t = 0$, the validity of this solution is restricted to $t \ll 1/s\varepsilon^2$.

In order to estimate the effect of nonlinearity in the full equation (6.6) for longer time scales, we introduce the transformation

$$a = \frac{\mu(t)}{2(\kappa\varepsilon)^{1/2}}, \quad \mu(t) = O(1), \tag{6.7}$$

which induces a balance between the lowest-order linear term and the dominant nonlinear terms. For the reduced equation we then obtain

$$\frac{\partial \mu}{\partial t} = \left(\frac{s\kappa w\varepsilon}{c_0}\right)\mu f(\mu) + O(s\varepsilon^2), \quad f(\mu) = 1 - \mu^2.$$

The next dominant nonlinear term in (6.6), which is proportional to $s\varepsilon^3 a^5$, is determined by matching at higher order. Then we obtain

$$f(\mu) = 1 - \mu^2 + \mu^4 + O(\mu^6),$$

which suggests that $f(\mu) = 1/(1 + \mu^2)$, valid for $\mu < 1$. Thus, we suppose that (6.7) transforms equation (6.6) into an equation of the form

$$\frac{\partial \mu}{\partial t} = \frac{s\kappa w\varepsilon}{c_0}\left(\frac{\mu}{1 + \mu^2}\right) + O(s\varepsilon^2). \tag{6.8}$$

The transformation (6.7) corresponds to

$$\sigma = \left(\frac{\varepsilon}{\kappa}\right)^{1/2}\mu(t), \tag{6.9}$$

where σ is the wave slope, with $\sigma = 2\varepsilon a$.

We recall that for a wave slope of $O(\varepsilon)$ the initial evolution of the wave is described by a Landau-type equation. When the above restriction is violated, however, this equation ceases to be valid. We also derived an alternative evolution equation of the form (6.8), valid for larger wave slope. The remainder term (which depends on μ as well) is indeed a higher-order correction term because of the condition $\mu < 1$. Since equation (6.8) is a transformed version of equation (6.6), including higher-order terms, it follows that, even for small wave slope, the former equation applies as well, where $\mu < 1$ implies that the remainder term is negligibly small if $\varepsilon \ll 1$. On the other hand, there is again an upper bound on the wave slope because, in view of (6.9),

$$\sigma < \left(\frac{\varepsilon}{\kappa}\right)^{1/2}. \tag{6.10}$$

We conclude that for $\varepsilon \ll 1$ the truncated form of equation (6.8) is valid for any initial wave slope and governs the evolution of the wave as long as condition (6.10) is satisfied. For $\varepsilon > 0.036$ the upper bound in (6.10) should be replaced by the threshold value $\sigma_c \approx 0.3$ for wave breaking.

Based on the evolution equation (6.8) we find that the effect of nonlinearity is measured by the single parameter μ, with (6.9) defined by

$$\mu = \sigma(\varepsilon/\kappa)^{-1/2}. \tag{6.11}$$

The parameter μ in (6.11) is now expressed in terms of the wave slope and the parameter ε. Making use of the Charnock (1955) relation, the latter parameter can be expressed in terms of the ratio u_*/c of the friction velocity to the phase velocity. For

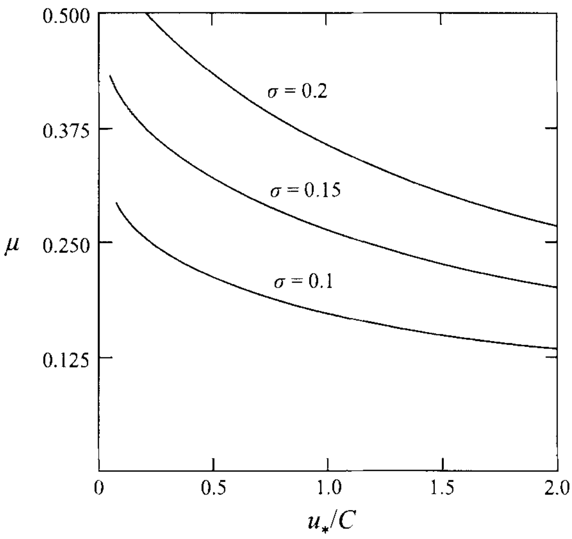

FIGURE 1. The parameter μ (as a measure of the effect of nonlinearity) against u_*/c for various wave slopes σ.

details the reader is referred to VDJ. In figure 1 the dependence of the parameter μ on u_*/c is sketched for various wave slopes σ. We deduce that the effect of nonlinearity increases with the wave slope and the wavelength. For high-frequency waves, nonlinear effects become noticeable for a wave slope exceeding 0.2, say. For lower frequencies, on the other hand, these effects become noticeable for smaller wave slope.

Equation (6.8) has a solution determined from

$$s\varepsilon t = \frac{c_0}{\kappa w}\left\{\log\left(\frac{\sigma}{\sigma_0}\right) + \frac{\kappa}{2\varepsilon}(\sigma^2 - \sigma_0^2)\right\}, \tag{6.12}$$

with $\sigma = \sigma_0$ at $t = 0$. As expected, nonlinearity increases the transition time from the initial wave slope $\sigma = \sigma_0$ to the occurrence of wave breaking. This is illustrated in figure 2, where the evolution with time of the wave slope is sketched for $\sigma_0 = 0.1$, $\varepsilon = 0.04$ and various c_0. The results are compared with those of linear theory.

7. The growth-rate parameter and the drag coefficient

The growth-rate parameter β is defined by

$$\beta = \frac{2}{\mid c_0 \mid}\ \frac{\partial a/\partial t}{a}, \tag{7.1}$$

which corresponds to the energy growth rate per radian (Miles 1957).

When expressed in terms of the wave slope, equation (6.6) implies that

$$\beta = \frac{2s\kappa w\varepsilon}{c_0 \mid c_0 \mid}\left\{1 + \frac{\varepsilon}{\kappa} - \frac{\kappa\sigma^2}{\varepsilon}\right\}, \quad \sigma = O(\varepsilon). \tag{7.2}$$

In VDJ, and also in Jacobs (1987), the reference height for the wind speed was chosen to be $1/k$, which corresponds to $\alpha = 1$ in (2.5). The expression (7.2) for the growth rate, on the other hand, is based on a value of α given by (6.5). According to Miles (1993), however, the parameter α should have the prescribed value $\alpha = 0.281$,

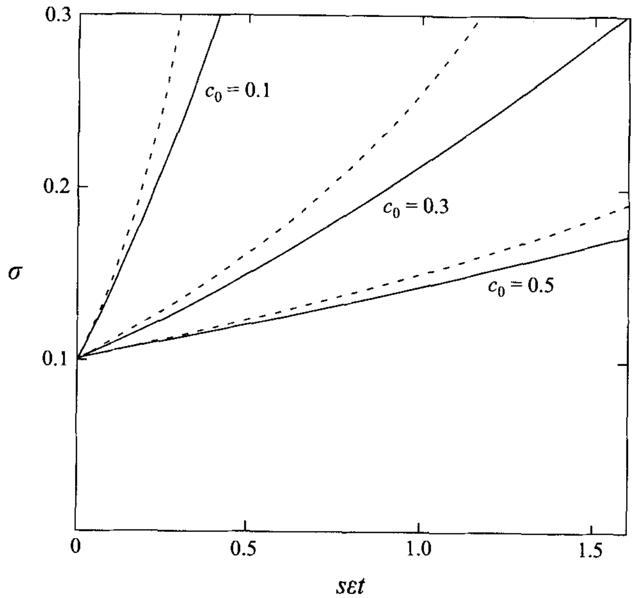

FIGURE 2. The evolution with time of the wave slope σ for various dimensionless phase speeds c_0; $\sigma_0 = 0.1$, $\varepsilon = 0.04$. Solid lines: nonlinear theory; dashed lines: linear theory.

which leads to a slightly different magnitude of the growth rate. However, we found that α is a free parameter of the order of unity. This discrepancy can be resolved as follows. From (2.5) and (2.6) it is found that $\varepsilon = \varepsilon(\alpha)$ and $c = c(\alpha)$, where c is the dimensionless phase velocity. From equation (6.3) we obtain

$$\beta = \frac{2s\kappa w\varepsilon}{c \mid c \mid} + O(s\varepsilon^2), \tag{7.3}$$

where $\varepsilon = \varepsilon(\alpha)$ and $c = c(\alpha)$, with arbitrary α. However, since $\varepsilon(1)$ and ε are the same at leading order (which applies to $c(1)$ and c as well), the difference between the results for $\alpha = 1$ and $\alpha \neq 1$ is absorbed into the remainder term in the expression (7.3) for β.

Expression (7.2) shows that the growth rate decreases with increasing wave slope, which implies that nonlinearity has a stabilizing effect. However, for a wave slope of the order of 0.1 this effect is rather small for high-frequency waves.

The validity of the analysis is retricted to slow waves because of the condition

$$\mid w \mid \gg \varepsilon, \tag{7.4}$$

with $w = (V - c)/V$, where c is the dimensional phase velocity (VDJ).

The (actual) wind speed $\bar{V}$ at elevation $z = z_1$ in the laboratory frame (scaled by the wind speed V, defined by (2.5)) is obtained from (4.8) and (4.13). In terms of the wave slope we find, with α given by (6.5),

$$\bar{V}(z_1) = 1 + \frac{\varepsilon}{\kappa} \log\left(\frac{z_1}{\alpha}\right) - \frac{\kappa w \sigma^2}{\varepsilon}, \quad \sigma = O(\varepsilon). \tag{7.5}$$

The (actual) drag coefficient C_d is defined by

$$C_d = (u_*/V_*)^2, \quad V_* = V\bar{V}, \tag{7.6}$$

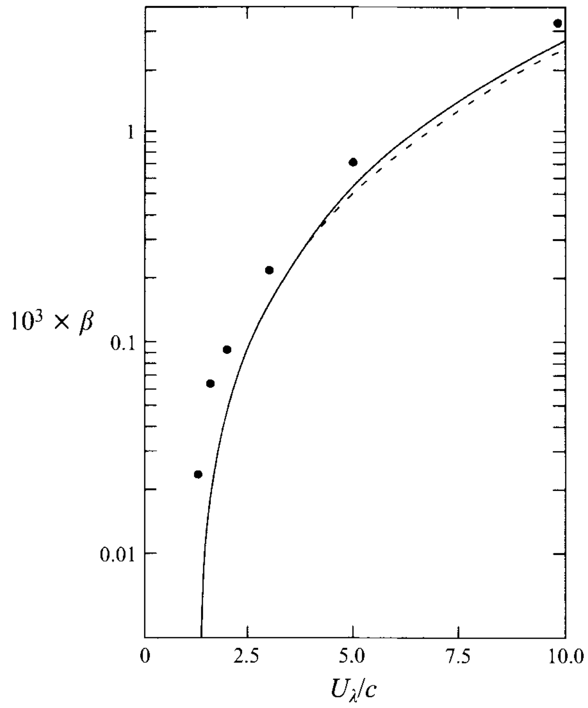

FIGURE 3. The growth-rate parameter (7.2) against U_λ/c (solid line), compared with the numerical results of Burgers & Makin (dots); $\kappa = 0.4$, $s = 0.0013$, $\varepsilon = 0.04$, $\sigma = 0.1$. Dashed line: linear theory.

where V_* is the dimensional wind speed at $z = z_1$. Combining (7.5) and (7.6) we obtain

$$C_d(z_1) = \varepsilon^2 \left\{ 1 + \frac{\varepsilon}{\kappa} \log\left(\frac{z_1}{\alpha}\right) - \frac{\kappa w \sigma^2}{\varepsilon} \right\}^{-2}. \tag{7.7}$$

The parameter ε corresponds to the square root of the drag coefficient $C_{df}(\alpha)$ for a flat water surface. Then (7.7) implies that the presence of a wave with phase speed smaller than the wind speed ($c_0 < 1$) increases the drag coefficient according to

$$C_d(\alpha) = C_{df}(\alpha) \left\{ 1 - \frac{\kappa w \sigma^2}{\varepsilon} \right\}^{-2}. \tag{7.8}$$

In figures 3 and 4 the growth-rate parameter (7.2) and the drag coefficient (7.7) are compared with the nonlinear, numerical results obtained by Burgers & Makin (1993). These authors used the same turbulence model. For comparison with their results we should take $\kappa = 0.4, s = 0.0013$, $\varepsilon = 0.041$ and $\sigma = 0.1$.

In figure 3 the growth-rate parameter (7.2), represented by a solid line, is sketched for various U_λ/C, where U_λ is the wind speed at an elevation of one wavelength, and C is the dimensional phase velocity. The numerical results of Burgers & Makin are represented by dots. For large U_λ/C, corresponding to high-frequency waves, the agreement with the numerical results is quite favourable. As expected on the basis of condition (7.4) for the validity of the analysis, the agreement rapidly worsens for smaller U_λ/C. The growth-rate parameter for the linear theory is represented by a dashed line.

In figure 4 the drag coefficient at elevation $z = \lambda$, obtained from (7.7) with $z_1 = 2\pi$, and represented by a solid line, is sketched for various U_λ/C. From a comparison

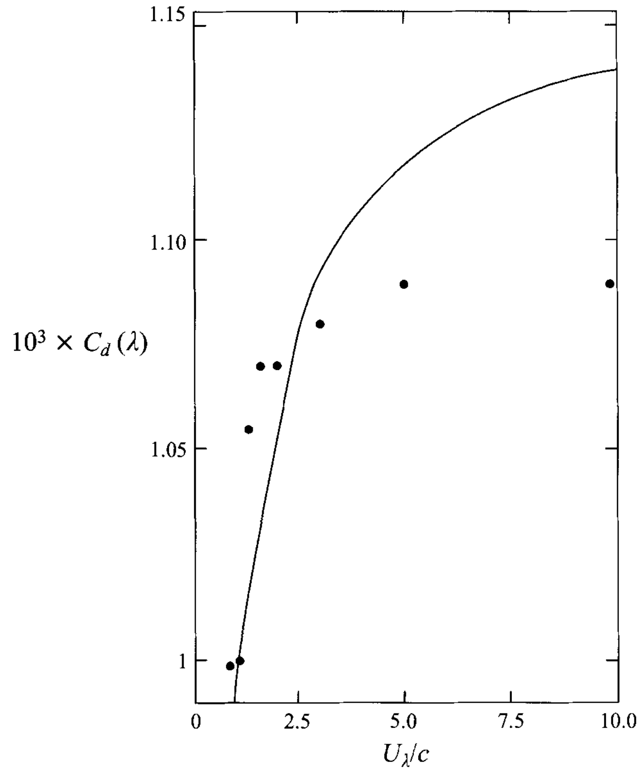

FIGURE 4. The drag coefficient (7.7), at elevation $z = \lambda$, against U_λ/c (solid line), compared with the numerical results of Burgers & Makin (dots); $\kappa = 0.4$, $s = 0.0013$, $\varepsilon = 0.04$, $\sigma = 0.1$.

with the numerical results of Burgers & Makin (represented by dots) we again find that the agreement is quite satisfactory for high-freqency waves.

8. Discussion of the results

The principal purpose of the present study was to investigate analytically the effect of nonlinearity on the generation of surface gravity waves by wind. In particular, we attempted to solve an initial-value problem for weakly nonlinear waves, where the initial conditions are prescribed by a single Stokes wave. The initial wave slope should be of the order of ε, where the small parameter ε corresponds to the square root of a characteristic drag coefficient.

The dynamics of the process of wave generation is governed by an evolution equation for the wave slope, obtained from the continuity condition for the normal stress at the air–water interface. The solution of this equation, which describes the change of the mean air flow with the growing or decaying wave, is uniquely determined by the initial wave slope.

Initially, and for sufficiently small wave slope, the evolution equation is approximately a Landau equation, which has an exact solution. In the course of time, however, the amplitude of the growing wave becomes so large that the latter equation, which is a truncation of the full evolution equation, ceases to be valid. This implies that its validity is restricted to sufficiently small time scales as well. By introduction of a suitable scale transformation, however, we derived an alternative evolution equation, which is valid for larger wave slope and larger time scales than the original Landau equation, and presumably describes the further evolution of the wave until the occurrence of wave breaking. Based on the alternative equation, we found that the relative effect of nonlinearity can be characterized by a single parameter, which contains the wave slope and the characteristic drag coefficient only. It turns out that this effect increases with increasing wave slope and decreasing wave frequency.

Based on the evolution equation we also derived a closed-form expression for the growth-rate parameter, which depends on the wave slope. It turns out that the wave growth rate decreases with increasing wave slope, which implies that nonlinearity reduces the rate of energy transfer from the mean air flow to the growing wave. The mean air flow decreases with increasing wave slope, which is consistent with energetic aspects of the mechanism of wave–mean flow interaction. Nonlinearity has a stabilizing effect, in agreement with results previously obtained by Fabrikant (1976) and Janssen (1982).

For a wave slope of the order of 0.1 the effect of nonlinearity on the growth rate of high-frequency waves is only small. For larger wave slope, of the order of 0.2 say, this effect becomes noticeable, especially for waves of lower frequency. The same applies to the effect of nonlinearity on the drag coefficient. This may differ considerably from the drag coefficient for flow over a flat water surface.

The validity of the present analysis is restricted to small phase speed and small characteristic drag coefficient. This was verified and confirmed by Jenkins (1992), who numerically solved the physically similar, linear problem of Jacobs (1987). The expression for the growth rate, obtained by the latter author on the basis of an asymptotic analysis, is the same as our expression in the limit of vanishing wave slope. The calculations of Jenkins imply that our result for the growth rate is generally smaller than his numerical result. However, the difference is small if the above restrictions are satisfied, in agreement with the asymptotic nature of the present analysis.

To some extent, this also explains why our result for the growth rate is generally smaller than the numerical result obtained by Burgers & Makin (1993), based on nonlinear calculations. This is especially the case for large phase speed. Within the above limitations, however, the agreement is favourable. The same applies when our expression for the drag coefficient is compared with the result of Burgers & Makin.

To describe the direct effects of turbulence, we introduced an eddy viscosity model, applied throughout the flow. It is expected that the incorporation of a physically more 'realistic' turbulence model (which has been, however, rather hypothetical up to now) will again lead to the rather straightforward calculations needed to solve the problem, because the methods applied remain the same. In this respect, it is also important to note that the lower-order solutions are independent of any closure assumptions, as was already found by Sykes (1980). Although we realize that some of the quantitative results will be different for a more realistic turbulence model, it is hoped that the principal results will not be affected significantly.

The author wishes to thank Peter Janssen for valuable discussions and useful comments on drafts of this paper.

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